

# Properties of bottomonium in a relativistic quark model

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## Abstract

The mass spectrum of  $b\bar{b}$  states has been obtained using the phenomenological relativistic quark model (RQM). The Hamiltonian used in the investigation has confinement potential and confined one gluon exchange potential (COGEP). In the frame work of RQM a study of M1 and E1 radiative decays of  $b\bar{b}$  states have been made. The weak decay widths in the spectator quark approximation have been estimated. An overall agreement is obtained with the experimental masses and decay widths.

*Key words:* relativistic quark model (RQM); radiative decay; confined one gluon exchange potential (COGEP);  $b\bar{b}$  states

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## 1. Introducing Bottomonium states

The bound state of a bottom quark  $b$  and its anti quark  $\bar{b}$  known as bottomonium, play an important role in the study of the strong interactions. The spectrum of bottomonium is known very well from the experiment and there are many different approaches to calculate it theoretically. Phenomenological potential models provide an accurate way of doing this. Bottomonium was first observed in the spectrum of  $\mu\mu^-$  pairs produced in 400 GeV proton–nucleus collisions at Fermilab. The  $b\bar{b}$  states were dicovered as spin triplet states  $\Upsilon(1S)$ ,  $\Upsilon(2S)$  and  $\Upsilon(3S)$  by E288 collaboration at Fermilab [1, 2].

The spin-singlet state  $\eta_b(1S)$  was discovered by the Babar Collaboration and they measured the mass of  $\eta_b(1S)$  to be  $M = 9388.9^{+3.1}_{-2.3}(\text{stat}) \pm 2.7(\text{syst})$  MeV[3]. In an another measurement the BaBar found the mass of  $\eta_b(1S)$  to be  $M = 9394^{+4.8}_{-2.3} \pm 2.7$  MeV. The CLEO collaboration measured the mass of  $\eta_b(1S)$  to be  $9391 \pm 6.6$  MeV[4]. The more precise measurement of mass of  $\eta_b(1S)$  state is done by the Belle collaboration which obtained a value of  $M = 9402.4 \pm 1.5 \pm 1.8$  MeV[5]. The  $\eta_b(2S)$  was successfully observed by the CLEO collaboration in  $\Upsilon(2S) \rightarrow \eta_b(2S)\gamma$  decays at a mass of  $9974.6 \pm 2.3 \pm 2.1$  MeV[6]. The Belle collaboration has reported a signal for  $\eta_b(2S)$  using the  $h_b(2P) \rightarrow \eta_b(2S)\gamma$  at a mass of  $9999.0 \pm 3.5^{+2.8}_{-1.9}$  MeV[5].

The BaBar collaboration first reported the evidence for the spin singlet P wave state  $h_b(1P)$  in the transition  $\Upsilon(3S) \rightarrow \pi^0 h_b(1P) \rightarrow \pi^0 \gamma \eta_b(1S)$  [7]. Later the Belle collaboration found  $h_b(1P)$  through  $\Upsilon(5S) \rightarrow h_b(1P)\pi^+\pi^-$  transition [8]. The measured masses of  $h_b(1P)$  and  $h_b(2P)$  were  $9898.25 \pm 1.06^{+1.03}_{-1.07}$  MeV and  $10259.76 \pm 0.64^{+1.43}_{-1.03}$  MeV respectively. The two triplet P-wave states  $\chi_{bJ}(2P)$  and  $\chi_{bJ}(1P)$  with  $J=0,1,2$  were discovered in radiative decays of the  $\Upsilon(3S)$  and  $\Upsilon(2S)$  respectively [9, 10, 11, 12]. The states  $\chi_{bJ}(nP)$  were produced in the proton - proton collisions at the LHC at  $\sqrt{s} = 7$  TeV and recorded by the ATLAS detector[13]. In addition to this, a new state  $\chi_{bJ}(3P)$  centred at a mass of  $10530 \pm 5 \pm 9$  MeV has been observed in both the  $\Upsilon(1S)\gamma$  and  $\Upsilon(2S)\gamma$  decay modes. This state was confirmed by the D0 collaboration which observed the  $\chi_{bJ}(3P)$  state in the  $\Upsilon(1S)\gamma$  final state with mass of  $10551 \pm 14 \pm 17$  MeV[14]. The LHCb collaboration has measured the mass of the  $\chi_{b1}(3P)$  to be  $10515^{+2.2}_{-3.9}(\text{stat})^{+1.5}_{-2.1}(\text{syst})$  MeV[15].

The spin triplet D-wave state  $\Upsilon(1^3D_2)$  was observed through the decay chain  $\Upsilon(3S) \rightarrow \gamma\gamma\Upsilon(1^3D_2) \rightarrow \gamma\gamma\pi^+\pi^-\Upsilon(1S)$ [16, 17]. The  $\Upsilon(3^3D_J)$  was discovered in the  $\pi^+\pi^-\Upsilon(1S)$  final state with mass  $M = 10164.5 \pm 0.8(\text{stat}) \pm 0.5(\text{syst})$  MeV[17]. The radial excitations of vector bottomonium family  $\Upsilon(4S)$ ,  $\Upsilon(10860)$  and  $\Upsilon(11020)$  were also observed[18, 19]. The two states  $X(10610)^\pm$  and  $X(10650)^\pm$  were observed by the Belle collaboration in the mass spectra of  $\pi^\pm\Upsilon(nS)$  ( $n=1,2,3$ ) and  $\pi^\pm h_b(nP)$  ( $n=1,2$ ) pairs[20]. These states do not fall into the usual scheme of mesons since they posses electrical charge. These states can be identified as molecular states, since these are very near to the  $BB^*$  and  $B^*B^*$  thresholds.

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QCD motivated potential models have played an important role in the understanding of quarkonium spectroscopy. These models have successfully predicted the bottomonium spectrum but over estimated the decay rates. In the present work we study bottomonium spectrum in detail using relativistic quark model.

The paper is organized in 4 sections. In sec. 2 we briefly review the theoretical background for relativistic model and the relativistic description of radiative decay widths. In sec. 3 we discuss the results and the conclusions are drawn in sec. 4 with a comparison to other models.

## 2. Phenomenological Understanding

### 2.1. The Relativistic Harmonic Model

Describing a bottomonium bound state problem, incorporating the elements of non perturbative low energy dynamics is a challenging problem. The two main approaches in this line have been effective field theories[21, 22, 23] and lattice gauge theories. Though the calculations away from threshold have been proven successful, the threshold region remain still tricky. And hence a relativistic potential model would be a befitting one for our study of bottomonium states. Essentially in all phenomenological relativistic QCD base quark models, the Hamiltonian for the quark system consists of a two body confinement potential and Coulomb like potential which takes care of relativistic effects. The Hamiltonian in our model has the confinement potential and a two body confined one gluon exchange potential(COGE) [24, 25, 26, 27].

The RHM equation is [28, 29, 30]:

$$(\alpha \cdot p + \gamma_0 M + \frac{1}{2}(1 + \gamma_0)A^2 r^2)\psi = E\psi \quad (1)$$

The Dirac wave function  $\psi$  is written as  $|\psi\rangle = N \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ , where  $\phi$  is the large componet and  $\chi$  is the small component for positive energy state solution in the non-relativistic limit. Using standard matrix form of  $\alpha$  and  $\beta$ , the Dirac equation is expressed in terms of the large and small components.

$$(A^2 r^2 + M)\phi + (\sigma \cdot p)\chi = E\phi \quad (2)$$

$$(\sigma \cdot p)\phi - (M + E)\chi = 0 \quad (3)$$

Hence, solution of the full Dirac equation  $\psi$  is written as

$$|\psi\rangle = N \begin{pmatrix} \phi \\ \frac{\sigma \cdot p}{E+M}\phi \end{pmatrix} \quad (4)$$

where  $\gamma_0$  is the Dirac matrix,  $M$  is a constant mass and  $A^2$  is the confinement strength.

with normalization constant

$$N = \sqrt{\frac{2(E+M)}{3E+M}} \quad (5)$$

$E$  is the eigenvalue of the single particle Dirac equation with interaction potential given by equation (??). We perform a unitary transformation to eliminate the lower component of  $\psi$ . With this transformation,  $\phi$  satisfies the 'harmonic oscillator' wave equation

$$\left( \frac{p^2}{E+M} + A^2 r^2 \right) \phi = (E - M)\phi \quad (6)$$

the eigenvalue of which is given by

$$E_N^2 = M^2 + (2N_{OSC} + 1)\Omega_N \quad (7)$$

where  $N_{OSC} = 2n+l$  is the oscillator quantum number and  $\Omega_N$  is the energy dependent oscillator size parameter given by

$$\Omega_N = A(E_N + M)^{1/2} \quad (8)$$

The total energy or the mass of the meson is obtained by adding the individual contributions of the quarks. The spurious centre of mass (CM) is corrected [31] by using intrinsic operators for the  $\sum_i r_i^2$  and  $\sum_i \nabla_i^2$  terms appearing in the Hamiltonian. This amounts to just subtracting the CM motion zero contribution from the  $E^2$  expression.

## 2.2. Confined One Gluon Exchange Potential

NRQM employing OGEP, which takes into consideration the confining effect of quarks on mesonic states, does not shed light on confinement of gluons. Hence one needs to incorporate confining effects of gluons on mesonic states since confined dynamics of gluons plays a decisive role in determining the hadron spectrum and in hadron-hadron interaction. There are various confinement models for the gluons. The effect of the confined gluons on the masses of mesons has been studied using the successful current confinement model (CCM). The confined gluon propagators (CGP) are derived in CCM to obtain the COGEP.

The central part of the COGEP is [24, 25, 26, 27]

$$V_{COGEP}^{cent}(\vec{r}) = \frac{\alpha_s N^4}{4} \lambda_i \cdot \lambda_j \left[ D_0(\vec{r}) + \frac{1}{(E+M)^2} [4\pi\delta^3(\vec{r}) - c^4 r^2 D_1(\vec{r})] \left[ 1 - \frac{2}{3} \vec{\sigma}_i \cdot \vec{\sigma}_j \right] \right] \quad (9)$$

where  $\lambda_i, \lambda_j$  are the color matrices,  $D_0(\vec{r})$  and  $D_1(\vec{r})$  are the propagators given by

$$D_0(\vec{r}) = \frac{\Gamma_{1/2}}{4\pi^{3/2}} c(cr)^{-3/2} W_{1/2; -1/4}(c^2 r^2) \quad (10)$$

$$D_1(\vec{r}) = \frac{\Gamma_{1/2}}{4\pi^{3/2}} c(cr)^{-3/2} W_{0; -1/4}(c^2 r^2) \quad (11)$$

Here  $\Gamma_{1/2} = \sqrt{\pi}$ ,  $W$ 's are Whittaker functions and  $c(\text{fm}^{-1})$  is a constant parameter which gives the range of propagation of gluons and is fitted in the CCM to obtain the glue-ball spectra and  $r$  is the distance from the confinement center.

The spin orbit part of COGEP is

$$V_{12}^{LS}(\vec{r}) = \frac{\alpha_s}{4} \frac{N^4}{(E+M)^2} \frac{\lambda_1 \cdot \lambda_2}{2r} \times \left[ [\vec{r} \times (\hat{P}_1 - \hat{P}_2) \cdot (\sigma_1 + \sigma_2)] (D'_0(\vec{r}) + 2D'_1(\vec{r})) + \right. \\ \left. [\vec{r} \times (\hat{P}_1 + \hat{P}_2) \cdot (\sigma_1 - \sigma_2)] (D'_0(\vec{r}) - D'_1(\vec{r})) \right] \quad (12)$$

where  $\hat{P}_1, \hat{P}_2$  are the momentum of quarks. The spin orbit term has been split into the symmetric  $(\sigma_1 + \sigma_2)$  and anti symmetric  $(\sigma_1 - \sigma_2)$  spin orbit terms.

The tensor part of the COGEP is,

$$V_{12}^{TEN}(\vec{r}) = -\frac{\alpha_s}{4} \frac{N^4}{(E+M)^2} \lambda_1 \cdot \lambda_2 \left[ \frac{D''_1(\vec{r})}{3} - \frac{D'_1(\vec{r})}{3r} \right] S_{12} \quad (13)$$

where

$$S_{12} = [3(\sigma_1 \cdot \hat{r})(\sigma_2 \cdot \hat{r}) - \sigma_1 \cdot \sigma_2] \quad (14)$$

The total potential is then

$$V(\vec{r}) = V_{COGEP}^{cent}(\vec{r}) + V_{12}^{LS}(\vec{r}) + V_{12}^{TEN}(\vec{r}) \quad (15)$$

We then obtain Hamiltonian by adding interaction potential given by equation 15 to the the 'harmonic oscillator' wave equation 6,

$$H = \sum_{i=1}^2 \frac{P_i^2}{E+M} - K_{CM} - \sum_{i < j} A^2 r_{ij}^2 \lambda_i \cdot \lambda_j + V(\vec{r}) \quad (16)$$

## 2.3. Radiative Decays

We consider two types of radiative transitions of the  $b\bar{b}$  meson:

### 2.3.1. Electric Dipole (E1) Transitions

The partial widths for electric dipole (E1) transitions between states  $^3S_1$  and  $^3P_J$  are given by

$$\Gamma_{(i \rightarrow f + \gamma)} = (2J' + 1) \frac{4}{3} Q_b^2 \alpha k_0^3 S_{if}^E |\mathcal{E}_{if}|^2 \quad (17)$$

where  $k_0$  is the energy of the emitted photon,

$$k_0 = \frac{m_i^2 - m_f^2}{2m_i} \text{ in relativistic model.}$$

$\alpha$  is the fine structure constant.  $Q_b = 1/3$  is the charge of the b quark in units of  $|e|$ .  $m_i$  and  $m_f$  are the masses of initial and final mesons. The statistical factor  $S_{if}^E = \max(l, l') \left\{ \begin{matrix} J & 1 & J' \\ l' & s & l \end{matrix} \right\}^2$ ,  $J, J'$  are the total angular momentum of initial and final mesons,  $l, l'$  are the orbital angular momentum of initial and final mesons and  $s$  is the spin of initial meson.

$$\mathcal{E}_{if} = \frac{3}{k_0} \int_0^\infty r^3 R_{nl}(r) R'_{nl}(r) dr \left[ \frac{k_0 r}{2} j_0 \left( \frac{k_0 r}{2} \right) - j_1 \left( \frac{k_0 r}{2} \right) \right] \quad (18)$$

is the radial overlap integral which has the dimension of length, with  $R_{nl}(r)$  being the normalized radial wave functions for the corresponding states.

### 2.3.2. Magnetic Dipole (M1) Transitions

The M1 partial decay width between S wave states is [32, 33, 34, 35, 36, 37, 38]

$$\Gamma_{i \rightarrow f \gamma} = \delta_{LL'} 4\alpha k_0^3 \frac{E_b(k_0)}{m_i} \left( \frac{Q_b}{m_b} + (-1)^{S+S'} \frac{Q_{\bar{b}}}{m_{\bar{b}}} \right)^2 (2S+1) \times (2S'+1)(2J'+1) \quad (19)$$

$$\left\{ \begin{matrix} S & L & J \\ J' & 1 & S' \end{matrix} \right\}^2 \left\{ \begin{matrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & S & S' \end{matrix} \right\}^2 \times \left[ \int_0^\infty R_{n'L'}(r) r^2 j_0(kr/2) R_{nL}(r) dr \right]^2$$

where  $\int_0^\infty dr R_{n'L'}(r) r^2 j_0(kr/2) R_{nL}(r)$  is the overlap integral for unit operator between the coordinate wave functions of the initial and the final meson states,  $j_0(kr/2)$  is the spherical Bessel function,  $m_b$  is the mass of bottom quark.  $J$  and  $J'$  are the total angular momentum of initial and final meson states respectively.  $L, L', S$  and  $S'$  are the orbital angular momentum and spin angular momentum of initial and final meson states respectively

## 3. Discussion on Results

### 3.1. Mass Spectra

The quark-antiquark wave functions in terms of oscillator wave functions corresponding to the relative and center of mass coordinates have been expressed here, which are of the form,

$$\Psi_{nlm}(r, \theta, \phi) = N' \left( \frac{r}{b} \right)^l L_n^{l+\frac{1}{2}} \left( \frac{r^2}{b^2} \right) \exp \left( -\frac{r^2}{2b^2} \right) Y_{lm}(\theta, \phi) \quad (20)$$

where  $N'$  is the normalising constant given by

$$|N'|^2 = \frac{2n!}{b^3 \pi^{1/2}} \frac{2^{[2(n+l)+1]}}{(2n+2l+1)!} (n+l)! \quad (21)$$

$L_n^{l+\frac{1}{2}}$  are the associated Laguerre polynomials.

The harmonic oscillator wave function allows the separation of the motion of the center of mass and has been widely used to classify the spectra of baryons and mesons [39, 40] and extending to nucleon-nucleon interaction is straight forward [41, 42, 26]. If the basic states are the harmonic oscillator wave functions, then it is straightforward to evaluate the matrix elements of few body systems such as mesons or baryons. Since the basic states are the products of the harmonic oscillator wave functions they can be chosen in a manner that allows the product wave functions to be expanded as a finite sum of the corresponding products for any other set of Jacobi coordinates. It is advantageous to use the Gaussian form since in the annihilation of quark-anti quark into lepton pairs, the amplitude of the emission or absorption processes depends essentially on the overlap of initial and final hadrons and hence the overlap depends only on the intermediate distance region of the spatial wave functions which can extend up to 0.5 fm. This intermediate region can be described by potentials that are similar in this region and hence harmonic oscillator wave functions are expected to reproduce emission and absorption processes quite well.

The five parameters used in our model are the mass of beauty quark  $m_b$ , the harmonic oscillator size parameter  $b$ , the confinement strength  $A^2$ , the CCM parameter  $c$  and the quark-gluon coupling constant  $\alpha_s$ . There are several papers in literature where the size parameter  $b$  is defined [43, 44]. The value of  $b$  is fixed by minimizing the expectation value of the Hamiltonian for the vector meson. To start with, we construct the

$5 \times 5$  Hamiltonian matrix for  $b\bar{b}$  states in the harmonic oscillator basis. The confinement strength  $A^2$  is fixed by the stability condition for variation of mass of the meson against the size parameter  $b$ .

$$\frac{\partial}{\partial b} \langle \psi | H | \psi \rangle = 0$$

The parameter  $c$  in CCM [24, 45, 46] was obtained by fitting the iota ( $1440 \text{ MeV}$ ) $0^{-+}$  as a digluon glue ball. To fit  $\alpha_s$ ,  $m_b$  we start with a set of reasonable values and diagonalize the matrix for  $b\bar{b}$  meson. Then we tune these parameters to obtain an agreement with the experimental value for the mass of  $b\bar{b}$  meson. In literature we find different sets of values for  $m_b$ , which are listed in Table 1.

Table 1: $m_b$ for various theoretical models (in MeV).					
Parameter	Ref. [47]	Ref. [48]	Ref. [49]	Ref. [50]	Ref. [51]
$m_b$	5174	5180	4880	4880	4880

The values of strong coupling constant  $\alpha_s$  in literature are listed in Table 2. The value of strong coupling constant ( $\alpha_s$ ) used is compatible with the perturbative treatment.

The mass spectrum has been obtained by diagonalizing the Hamiltonian in a large basis of  $5 \times 5$  matrix. The

Table 2: $\alpha_s$ for various theoretical models.					
Parameter	Ref. [52]	Ref. [51]	Ref. [53]	Ref. [54]	Ref. [55]
$\alpha_s$	0.21	0.265	0.357	0.361	0.391

calculation clearly indicates that masses for both pseudo scalar and vector mesons converge to the experimental values when the diagonalization is carried out in a larger basis. In our earlier work also, we had come to the similar conclusion while investigating light meson spectrum [29, 56]. The diagonalization of the Hamiltonian matrix in a larger basis leads to the lowering of the masses and justifies the perturbative technique to calculate the mass spectrum. The calculation clearly indicates that when diagonalization is carried out in a larger basis convergence is achieved both for pseudo-scalar mesons and vector mesons to the respective experimental values. We use the following set of parameter values.

$$m_b = 4770.0 \text{ MeV}; b = 0.35 \text{ fm}; \quad \alpha_s = 0.2 \text{ to } 0.3; \quad A^2 = 220 \text{ MeV fm}^{-2}; \quad c = 1.72 \text{ fm}^{-1} \quad (22)$$

The calculated masses of the  $b\bar{b}$  states after diagonalization are listed in Table 3. Our calculated mass value for  $\eta_b(1S)$  is  $9399 \text{ MeV}$  which agrees with the experimental value  $9398.0 \pm 3.2 \text{ MeV}$  [57] and for  $\Upsilon(1S)$  is  $9460.30 \pm 0.26$ .  $J/\psi(1S)$  is heavier than  $\eta_b(1S)$  by  $61 \text{ MeV}$ . This difference is justified by calculating the  $^3S_1 - ^1S_0$  splitting of the ground state which is given by

$$M(^3S_1) - M(^1S_0) = \frac{32\pi\alpha_s |\psi(0)|^2}{9m_b^2} \quad (23)$$

The experimental value of hyperfine mass splitting for the ground state is given by [57]

$$M(^3S_1) - M(^1S_0) = 62.3 \pm 3.2 \text{ MeV} \quad (24)$$

The hyperfine mass splitting calculated in our model is in good agreement with both experimental data [57] and lattice QCD result  $(60.3 \pm 7.7) \text{ MeV}$  [58]. The EFTs have predicted a hyperfine mass splitting of  $41 \pm 11_{-8}^{+9} \text{ MeV}$  [59] which is lower than the experimental value.

We predict a mass of  $10001.12 \text{ MeV}$  for the  $\eta_b(2S)$  state which is in good agreement with the experimental value  $9999.0 \pm 3.5 \text{ MeV}$  [57]. The calculated mass of spin triplet state ( $\Upsilon(2S)$ ) of  $\eta_b(2S)$  is  $10024.50 \text{ MeV}$  which is found to be in good agreement with the experimental value  $10023.26 \pm 0.31 \text{ MeV}$ . The hyperfine mass splitting for the  $2S$  state  $\Delta_{hf}(\eta_b(2S)) = 23.38 \text{ MeV}$  is in good agreement with both the experimental one  $\Delta_{hf}(\eta_b(2S)) = 24.3_{-4.5}^{+4.0} \text{ MeV}$  [57] and with the lattice QCD results  $\Delta_{hf}(\eta_b(2S)) = (23.5 - 28.0) \text{ MeV}$  [58]. The predicted mass values for  $\eta_b(3S)$  and  $\Upsilon(3S)$  states in our model are  $10375.96 \text{ MeV}$  and  $10392.00 \text{ MeV}$  respectively. The corresponding hyperfine mass splitting is  $\Delta_{hf}(\eta_b(3S)) = 16.04 \text{ MeV}$ . We have also calculated masses of higher radially excited states of  $\eta_b$  and its spin triplet state  $\Upsilon$ . The mass of spin singlet P wave state  $h_b(1P)$  calculated in our model is compatible with the experimental one. The masses of spin triplet states  $\chi_{bJ}$  are in good agreement with experimental values of masses of these states. Our prediction for masses of higher excited P wave states are  $10\text{-}20 \text{ MeV}$  higher than the experimental values and other model calculations. Some of the higher excited states are  $50\text{-}100 \text{ MeV}$  heavier in our model.

Table 3: **Bottomonium mass spectrum (in MeV).**

State $n \ ^{2S+1}L_J$	This work	Experiment[57]	Ref.[60]	Ref. [61]	Ref. [62]	Ref. [63]	Ref.[64]
$1 \ ^1S_0$	9399.74	$9398.0 \pm 3.2$	9455	9402	9391	9391.8	9394.5
$1 \ ^3S_1$	9460.77	$9460.30 \pm 0.26$	9502	9465	9489	9460.3	9459.2
$1^3P_0$	9856.75	$9859.44 \pm 0.42$	9855	9847	9849	9875.3	9850.0
$1^1P_1$	9898.95	$9899.3 \pm 1.0$	9879	9882	9885	9915.5	9884.4
$1^3P_1$	9903.73	$9892.78 \pm 0.26$	9874	9876	9879	9906.8	9878.4
$1^3P_2$	9919.21	$9912.21 \pm 0.26$	9886	9897	9900	9929.6	9897.1
$2 \ ^1S_0$	10001.12	$9999.0 \pm 3.5$	9990	9976	9980	10004.9	9982.6
$2 \ ^3S_1$	10024.50	$10023.26 \pm 0.31$	10015	10003	10022	10026.2	10012.1
$1^3D_1$	10151.14	-	10117	10138	10112	10138.1	10132.8
$1^1D_2$	10160.69	-	10123	10148	10122	10145.5	-
$1^3D_2$	10163.48	-	10122	10147	10121	10144.6	10138.3
$1^3D_3$	10170.97	-	10127	10155	10127	10149.3	-
$2^3P_0$	10244.15	$10232.5 \pm 0.4$	10221	10226	10226	10227.9	10233.6
$2^1P_1$	10269.15	$10259.8 \pm 0.5$	10240	10250	10247	10259.1	10262.7
$2^3P_1$	10275.77	$10255.46 \pm 0.22$	10236	10246	10244	10252.4	10257.7
$2^3P_2$	10287.14	$10268.65 \pm 0.22$	10246	10261	10257	10270.1	10273.9
$3 \ ^1S_0$	10375.96	-	10330	10336	10338	10337.9	10358.6
$3 \ ^3S_1$	10392.00	$10355.2 \pm 0.5$	10349	10354	10358	10351.9	10379.9
$2^3D_1$	10468.44	-	10414	10441	-	10420.4	10453.9
$2^1D_2$	10478.94	-	10419	10450	-	10427.98	-
$2^3D_2$	10479.40	-	10418	10449	-	-	-
$2^3D_3$	10486.20	-	10422	10455	-	-	-
$3^1P_1$	10560.66	-	10516	10541	10591	-	10568.5
$3^3P_0$	10566.17	-	10500	10522	10495	-	10540.2
$3^3P_1$	10589.96	$10515.7 \pm 2.2$	10513	10538	10580	-	10563.9
$3^3P_2$	10599.93	-	10521	10550	10578	-	10576.2
$4 \ ^1S_0$	10692.09	-	-	10623	-	-	-
$4 \ ^3S_1$	10704.51	$10579.4 \pm 1.2$	10607	10635	-	-	10718.0
$3^3D_1$	10757.06	-	10653	10698	-	-	10752.5
$3^1D_2$	10766.28	-	10658	10706	-	-	-
$3^3D_2$	10766.51	-	10657	10705	-	-	-
$3^3D_3$	10772.55	-	10660	10711	-	-	-
$4^1P_1$	10751.68	-	-	10790	-	-	-
$4^3P_0$	10857.83	-	-	10775	-	-	-
$4^3P_1$	10876.06	-	-	10788	-	-	-
$4^3P_2$	10882.90	-	10744	10798	-	-	-
$5 \ ^1S_0$	10980.94	-	-	10869	-	-	-
$5 \ ^3S_1$	10990.94	$10876 \pm 11$	10818	10878	-	-	10997.1
$4^3D_1$	11028.52	-	-	10928	-	-	11031.9
$4^1D_2$	11036.36	-	-	10935	-	-	-
$4^3D_2$	11036.38	-	-	10934	-	-	-
$4^3D_3$	11041.45	-	-	10939	-	-	-
$5^3P_0$	11133.06	-	-	10798	-	-	-
$5^3P_1$	11146.87	-	-	11004	-	-	-
$5^3P_2$	11153.78	-	-	11022	-	-	-
$5^1P_1$	10943.12	-	-	11016	-	-	-
$5^3D_1$	11287.23	-	-	-	-	-	11268.9
$5^1D_2$	11293.53	-	-	-	-	-	-
$5^3D_2$	11293.48	-	-	-	-	-	-
$5^3D_3$	11297.86	-	-	-	-	-	-
$6 \ ^1S_0$	11255.33	-	-	11097	-	-	-
$6 \ ^3S_1$	11308.41	$11019 \pm 8$	10995	11102	-	-	11232.3

### 3.2. Radiative Decays

Radiative decays of excited bottomonium states are the powerful tools which can be used to study the internal structure of  $b\bar{b}$  states and they provide a good test for the predictions from the various models. Radiative transitions between different bottomonium states can be used to discover new bottomonium states experimen-

tally.

The possible  $E1$  decay modes have been listed in Table 4 and the predictions for  $E1$  decay widths are given. Also our predictions have been compared with other theoretical models. Most of the predictions for  $E1$  transitions are in qualitative agreement. However, there are some differences in the predictions due to differences in phase space arising from different mass predictions and also from the wave function effects particularly from relativization procedure. We find our results are compatible with other theoretical model values for most of the channels. Relativistic corrections to the wave functions tend to reduce the  $E1$  transition widths for most of the channels. We find that the  $E1$  transitions  $3S \rightarrow 1P$  are suppressed compared to other  $E1$  transitions and it is interesting to see that the  $E1$  transition rates for  $3S \rightarrow 1P$  are zero in the nonrelativistic case[65, 66]. We find that these  $E1$  transitions are very sensitive to relativistic effects.

Table 4: **E1 transition rates of bottomonium.**

Transition	$k_0$ MeV	This work keV	Experiment[57] keV	Ref. [34] keV	Ref. [67] keV	Ref. [61] keV	Ref.[60]
$1^3P_0 \rightarrow 1^3S_1\gamma$	388.02	22.046	-	22.1	22.1	23.8	28.07
$1^3P_1 \rightarrow 1^3S_1\gamma$	433.05	29.754	-	27.8	27.3	29.5	35.66
$1^3P_2 \rightarrow 1^3S_1\gamma$	447.846	32.567	-	31.6	31.2	32.8	39.15
$1^1P_1 \rightarrow 1^1S_0\gamma$	428.481	28.913	-	41.8	37.9	35.7	43.66
$2^3S_1 \rightarrow 1^3P_0\gamma$	166.346	1.248	$1.21 \pm 0.16$	1.29	1.15	0.91	1.09
$2^3S_1 \rightarrow 1^3P_1\gamma$	120.042	0.479	$2.20 \pm 0.22$	2.0	1.87	1.63	1.84
$2^3S_1 \rightarrow 1^3P_2\gamma$	104.73	0.320	$2.28 \pm 0.22$	2.04	1.88	1.88	2.08
$2^1S_0 \rightarrow 1^1P_1\gamma$	101.647	0.292	-	1.99	4.17	2.48	2.85
$2^3P_0 \rightarrow 1^3S_1\gamma$	753.427	2.584	-	10.9	6.69	2.5	5.44
$2^3P_1 \rightarrow 1^3S_1\gamma$	782.68	3.284	-	12.0	7.31	5.5	9.13
$2^3P_2 \rightarrow 1^3S_1\gamma$	793.178	3.569	-	12.7	7.74	8.4	11.38
$2^1P_1 \rightarrow 1^1S_0\gamma$	832.606	4.822	-	-	-	13.0	14.90
$2^3P_0 \rightarrow 2^3S_1\gamma$	217.287	6.695	-	9.17	9.90	10.9	12.80
$2^3P_1 \rightarrow 2^3S_1\gamma$	248.197	9.729	-	12.4	13.7	13.3	15.89
$2^3P_2 \rightarrow 2^3S_1\gamma$	259.287	14.643	-	14.5	16.8	14.3	17.5
$2^1P_1 \rightarrow 2^1S_0\gamma$	264.532	15.476	-	19.0	37.9	14.1	17.60
$3^3S_1 \rightarrow 2^3P_0\gamma$	146.798	6.792	$1.19 \pm 0.16$	1.35	1.67	1.03	1.21
$3^3S_1 \rightarrow 2^3P_1\gamma$	115.580	3.385	$2.56 \pm 0.34$	2.20	2.74	1.91	2.13
$3^3S_1 \rightarrow 2^3P_2\gamma$	104.330	2.505	$2.66 \pm 0.41$	2.40	2.80	2.30	2.56
$3^1S_0 \rightarrow 2^1P_1\gamma$	106.260	2.644	-	2.10	-	1.7	2.60
$3^3S_1 \rightarrow 1^3P_0\gamma$	521.46	1.152	$0.054 \pm 0.08$	0.0001	0.03	0.01	0.15
$3^3S_1 \rightarrow 1^3P_1\gamma$	476.799	0.641	$0.018 \pm 0.001$	0.008	0.09	0.05	0.16
$3^3S_1 \rightarrow 1^3P_2\gamma$	462.035	0.520	$0.201 \pm 0.32$	0.015	0.13	0.45	0.0827
$3^1S_0 \rightarrow 1^1P_1\gamma$	466.045	0.551	-	0.007	-	1.30	0.0084

The  $M1$  transitions contribute little to the total decay widths of bottomonium states and are weaker than the  $E1$  transitions of bottomonium states.  $M1$  transitions have been used to observe the spin-singlet states[61]. Allowed  $M1$  transitions correspond to triplet-singlet transitions between S-wave states of the same  $n$  quantum number, while hindered  $M1$  transitions are either triplet-singlet or singlet-triplet transitions between S-wave states of different  $n$  quantum numbers.

In order to calculate decay rates of hindered transitions we need to include relativistic corrections. There are three main types of corrections: relativistic modification of the non relativistic wave functions, relativistic modification of the electromagnetic transition operator, and finite-size corrections. In addition to these there are additional corrections arising from the quark anomalous magnetic moment. Corrections to the wave function that give contributions to the transition amplitude are of two categories. Firstly from higher order potential corrections, which are distinguished as a) Zero recoil effect and b) recoil effects of the final state meson and secondly from colour octet effects. The colour octet effects are not included in potential model formulation and are not considered so far in radiative transitions.

The  $M1$  transition rates of bottomonium states have been calculated using equation (19). The resulting  $M1$  radiative transition rates of these states are presented in Table 5. In this table we give calculated values for decay rates of  $M1$  radiative transition in comparison with the other relativistic and non relativistic quark models. We see from these results that the relativistic effects play a very important role in determining the bottomonium  $M1$  transition rates. The relativistic effects reduce the decay rates of allowed transitions and increase the rates of hindered transitions. The  $M1$  transition rates calculated in our model agree well with the

values predicted by other theoretical models.

Table 5: **M1 transition rates of bottomonium.**

Transition	$k_0(MeV)$	$\Gamma$ eV	$\Gamma$ eV	$\Gamma^{NR}$ eV	$\Gamma$ eV	$\Gamma$ eV	$\Gamma$ eV
		This work	Experiment[57]	Ref.[51]	Ref.[51]	Ref.[61]	Ref.[60]
$\Upsilon(1S) \rightarrow \eta_b(1S)\gamma$	60.833	10.65	-	12.2	9.7	10.0	9.34
$\Upsilon(2S) \rightarrow \eta_b(2S)\gamma$	23.352	0.603	-	1.50	1.6	0.59	0.58
$\Upsilon(3S) \rightarrow \eta_b(3S)\gamma$	16.027	0.195	-	0.8	0.9	0.25	0.658
$\Upsilon(2S) \rightarrow \eta_b(1S)\gamma$	605.291	31.47	$12.47 \pm 4.90$	1.3	1.3	81	56.5
$\eta_b(2S) \rightarrow \Upsilon(1S)\gamma$	525.752	12.16	-	2.5	2.4	68	45.0
$\Upsilon(3S) \rightarrow \eta_b(1S)\gamma$	944.887	3.56	$10.36 \pm 1.70$	3.1	2.5	60	57.0
$\eta_b(3S) \rightarrow \Upsilon(1S)\gamma$	874.828	1.60	-	7.1	5.8	74.0	51.0
$\Upsilon(3S) \rightarrow \eta_b(2S)\gamma$	383.528	4.47	$12.59 \pm 1.14$	0.1	0.2	190	11.0
$\eta_b(3S) \rightarrow \Upsilon(2S)\gamma$	345.507	2.19	-	0.2	0.4	9.1	9.20

#### 4. Conclusions

The bottomonium mass spectrum and electromagnetic transitions are investigated by adopting the quark-antiquark potential consisting of the confined one-gluon-exchange and the Lorentz scalar plus vector harmonic oscillator confinement potentials. We perform a nonperturbative calculations with Hamiltonian including spin-independent and dependent potentials.

The bottomonium spectrum predicted by our quark model is in good agreement with the experimental data. We have calculated a large number of electromagnetic decay widths showing that our results are in reasonable agreement with the other model calculations in most of the cases. The hyperfine mass splitting between the singlet and triplet 1S as well as 2S states are consistent with the experimental data and also with lattice QCD results. We have also predicted radially excited states of  $\eta_b$  and their triplet states. The masses of P wave states calculated in our model are in good agreement with the experimental values and other theoretical models. Some of the higher excited states are 50-100 MeV heavier in our model. This may be due to the fact that coupled channel effects are significant in these states.

Radiative decays are the dominant decay modes of the  $b\bar{b}$  excited states having widths of about a fraction of MeV. In order to understand the  $b\bar{b}$  spectrum and distinguish exotic states from conventional ones, it is very essential that the masses and the radiative decay widths of  $b\bar{b}$  states are accurately determined. The calculated M1 transition rates reasonably agree with the other theoretical model predictions as listed in table 5. It is clearly seen in this calculation that the relativistic effects play an important role in determining the M1 radiative transition rates, since the hindered transition rates are zero due to the wave function orthogonality in the NRQM formalism. The inclusion of relativistic effects enhances the non-relativistically-hindered radiative transition rates, predicting them larger than the allowed ones by an order of magnitude. It is a good example for the importance of relativity, even for some properties of heavy mesons.

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